

Indecomposable Fusion Products

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Abstract

We analyse the fusion products of certain representations of the Virasoro algebra for $c = -2$ and $c = -7$ which are not completely reducible. We introduce a new algorithm which allows us to study the fusion product level by level, and we use this algorithm to analyse the indecomposable components of these fusion products. They form novel representations of the Virasoro algebra which we describe in detail.

We also show that a suitably extended set of representations closes under fusion, and indicate how our results generalise to all $(1, q)$ models.

1 Introduction

Recently it has become apparent that there exist conformal field theories whose correlation functions exhibit logarithmic behaviour. The presence of the logarithmic terms has been interpreted to signal the appearance of a new type of conformal operator, now usually called *logarithmic operator*. Models with such logarithmic operators include the WZNW model on the supergroup $GL(1, 1)$ [1], the $c = -2$ model [2], gravitationally dressed conformal field theories [3] and some critical disordered models [4]. They are believed to be important for the description of certain statistical models, in particular in the theory of (multi)critical polymers [5, 6, 7] and percolation [8]. There have also been suggestions that some of these logarithmic operators might correspond to normalisable zero modes for string backgrounds [9].

It was already realised in [2] that the logarithmic operators are not eigenstates of the scale generator L_0 , but that, under the action of L_0 , they form a Jordan cell with another field

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of the same conformal dimension. In particular, this means that the scale generator L_0 is not diagonalisable, and that the space of states of the theory is not a direct sum of irreducible Virasoro highest weight representations. This has raised the question of how to understand these novel representations of the Virasoro algebra [6, 7], and it is one of the aims of this paper to give a detailed description of a certain class of such indecomposable representations.

The way in which the logarithmic operators appear in the theory is as follows. It is well known that the two- and three-point functions of a (chiral) conformal field theory are determined up to constants by the conformal symmetry, and that every four-point function is fixed up to an *a priori* undetermined function $f(x)$ of one (complex) variable, the cross-ratio of the four coordinates. There is a subclass of theories (the so-called *degenerate* theories), for which certain descendents of the fields decouple, and this gives rise to differential equations for the undetermined function $f(x)$. The differential equations have typically (regular) singular points at $x = 0, 1$ and ∞ , and we can thus solve them by a power series expansion about any one of the singular points. In general, this will give all (different) solutions to the differential equation, unless two of the leading powers differ by an integer. Then there will sometimes be one (or more) solutions which involve a logarithmic term. Typically we cannot ignore these logarithmic solutions, as this would destroy the crossing symmetry of the conformal field theory.

The fields of a conformal field theory can be identified with states in a representation of the symmetry algebra, and the product of fields, the so-called *fusion*, can be understood as some kind of tensor product of these representations (which is again a representation of the symmetry algebra) [10, 11]. The different solutions to the differential equation for $f(x)$ (near the singular point $x = 0$, say) correspond then to the different subrepresentations which are contained in the fusion product (of the two fields whose distance vanishes at $x = 0$). In the usual case, the action of L_0 on this tensor product is diagonalisable, and the product can be decomposed into a direct sum of irreducible representations. In the logarithmic case, however, the action of L_0 is no longer diagonalisable, and the product contains reducible but indecomposable representations [11].

In this paper we want to study the indecomposable representations which are contained in the fusion products of certain highest weight representations of the Virasoro algebra for $c = -2$ and $c = -7$. We introduce a new algorithm which allows us to study the fusion product up to any (finite) level while only considering finite dimensional vector spaces. We then use this algorithm to study the fusion product level by level, and we can thus unveil the structure of the indecomposable representations.

It turns out that only certain types of indecomposable representations appear in the various fusion products, and we describe them in detail. These representations are characterised by one parameter which we determine explicitly for a number of cases. Similar representations have also been investigated independently by Rohsiepe [12].

We analyse the fusion of these indecomposable representations, and find that a certain set of representations, containing all highest weight representations and some indecomposable representations, is again closed under fusion. Finally, we indicate how some of our results generalise naturally to all $(1, q)$ models.

The paper is organised as follows. In Section 2, we introduce our algorithm, and show that it always terminates for the case of the Virasoro algebra. We then recall in Section 3, what is known about the fusion of highest weight representations of the Virasoro algebra, and explain why it is natural to expect that indecomposable representations appear in certain fusion products of the $(1, q)$ models. We describe the structure of the indecomposable representations in Section 4, and give a conjecture for the general fusion rules. In Section 5 we present the explicit results of our calculations which provide supportive evidence for these conjectures. In Section 6, we make some prospective remarks, and in the appendix we give explicit details of one specific example to illustrate our method.

2 Fusion and tensor products: The algorithm

Let us start by introducing some notation. We denote by \mathcal{A} the chiral algebra, *i.e.* the algebra generated by the modes of the holomorphic fields, and fix our convention as in [13], so that the modes of a field S of conformal weight h are given as

$$S(w) = \sum_{l \in \mathbb{Z}+h} w^{l-h} S_{-l}. \quad (2.1)$$

We also assume, as is usual in conformal field theory, that one of the fields is the stress-energy tensor $L(z)$ of weight 2, whose modes L_n satisfy the Virasoro algebra.

Given two representations of \mathcal{A} , \mathcal{H}_1 and \mathcal{H}_2 , and two points $z_1, z_2 \in \mathbb{C}$ in the complex plane, the fusion tensor product can be defined by the following construction [10]. First we consider the product space $(\mathcal{H}_1 \otimes \mathcal{H}_2)$ on which two different actions of the chiral algebra are given by the two comultiplication formulae [14]

$$\begin{aligned} \Delta_{\zeta, z}(S_n) = \tilde{\Delta}_{\zeta, z}(S_n) = & \sum_{m=1-h}^n \binom{n+h-1}{m+h-1} \zeta^{n-m} (S_m \otimes \mathbb{1}) \\ & + \varepsilon_1 \sum_{l=1-h}^n \binom{n+h-1}{l+h-1} z^{n-l} (\mathbb{1} \otimes S_l), \end{aligned} \quad (2.2)$$

$$\begin{aligned} \Delta_{\zeta, z}(S_{-n}) = & \sum_{m=1-h}^{\infty} \binom{n+m-1}{n-h} (-1)^{m+h-1} \zeta^{-(n+m)} (S_m \otimes \mathbb{1}) \\ & + \varepsilon_1 \sum_{l=n}^{\infty} \binom{l-h}{n-h} (-z)^{l-n} (\mathbb{1} \otimes S_{-l}), \end{aligned} \quad (2.3)$$

and

$$\begin{aligned} \tilde{\Delta}_{\zeta, z}(S_{-n}) = & \sum_{m=n}^{\infty} \binom{m-h}{n-h} (-\zeta)^{m-n} (S_{-m} \otimes \mathbb{1}) \\ & + \varepsilon_1 \sum_{l=1-h}^{\infty} \binom{n+l-1}{n-h} (-1)^{l+h-1} z^{-(n+l)} (\mathbb{1} \otimes S_l), \end{aligned} \quad (2.4)$$

where in (2.2) we have $n \geq 1 - h$, in (2.3, 2.4) $n \geq h$, and ε_1 is ∓ 1 according to whether the left-hand vector in the tensor product and the field S are both fermionic or not.³ The fusion tensor product is then defined as the quotient of the product space by all relations which come from the equality of Δ_{z_1, z_2} and $\tilde{\Delta}_{z_1, z_2}$

$$(\mathcal{H}_1 \otimes \mathcal{H}_2)_f := (\mathcal{H}_1 \otimes \mathcal{H}_2) / (\Delta_{z_1, z_2} - \tilde{\Delta}_{z_1, z_2}). \quad (2.5)$$

It has been shown for a number of examples that this definition reproduces the known restrictions for the fusion rules [10, 14].

In these calculations, the ring-like nature of the fusion product was not really used, and only certain necessary conditions for the fusion rules were derived. In this paper we want to introduce (and use) a more refined method for the study of the fusion product. We shall define a family of typically finite-dimensional quotient spaces which provide a filtration for the whole space. The action of the chiral algebra can be studied on all of these spaces, and it will turn out that essentially the whole structure of the fusion product can be understood by considering a finite number of them. This analysis is a natural generalisation of some ideas of Werner Nahm [11].

To define this filtration let us introduce some more notation. The chiral algebra \mathcal{A} contains two rather natural subalgebras. Firstly, there is the algebra which is generated by the negative modes which annihilate the vacuum

$$\mathcal{A}_-^0 := \langle S_{-n} | 0 < n < h(S) \rangle, \quad (2.6)$$

and secondly, the algebra generated by the negative modes which do not annihilate the vacuum

$$\mathcal{A}_{--} := \langle S_{-n} | n \geq h(S) \rangle. \quad (2.7)$$

We also denote by \mathcal{A}_- and \mathcal{A}_+ the algebras generated by all negative and positive modes, respectively.

Let \mathcal{H} be a representation of the chiral algebra \mathcal{A} . We define the *special subspace* of \mathcal{H} as the quotient space⁴ [11]

$$\mathcal{H}^s := \mathcal{H} / \mathcal{A}_{--} \mathcal{H}. \quad (2.8)$$

If \mathcal{H} is a highest weight representation, *i.e.* if \mathcal{H} is generated by the action of \mathcal{A}_- from a highest weight vector ψ which satisfies $\mathcal{A}_+ \psi = 0$, then it is always possible to realise the special subspace as a quotient space of $\mathcal{A}_-^0 \psi$. If not explicitly mentioned otherwise, we shall use this convention in the following. For the case of the Virasoro algebra (to which we shall restrict ourselves for a large part of the paper), there exists only one realisation of the special subspace as a subspace of $\mathcal{A}_-^0 \psi$.

In the following we shall only consider *quasirational* representations, *i.e.* representations for which the dimension of the special subspace is finite. As shown in [11], the fusion product of a quasirational representation with a highest weight representation contains at most finitely many (irreducible) subrepresentations.

³The second formula differs from the one given in [14] by a different ε factor. There the two comultiplication formulae were evaluated on different branches; this is corrected here.

⁴We should note that “the special subspace” is not defined as a subspace.

The special subspace is a very important concept for the analysis of the fusion product, but it has the rather important drawback that it does not carry a representation of \mathcal{A}_-^0 or even the zero modes. (For example, in the case of the W_3 algebra [15], the action of W_0 is not defined on \mathcal{H}^s , as $[W_0, L_{-2}] = 4W_{-2}$.) It is therefore useful to introduce a different set of quotient spaces. We define the subalgebra which is generated by all products of modes whose L_0 grading is greater or equal to n ,

$$\mathcal{A}_n := \langle \prod_{j=1}^m S_{-l_j}^{k_j} \mid \sum_{j=1}^m l_j \geq n \rangle, \quad (2.9)$$

and define then a filtration of \mathcal{H} as the family of quotient spaces

$$\mathcal{H}^n := \mathcal{H} / \mathcal{A}_{n+1} \mathcal{H}. \quad (2.10)$$

In particular, for a highest weight representation, \mathcal{H}^0 can be naturally identified with the space of highest weight states, and \mathcal{H}^n with the space of all descendents up to level n . However, the advantage of this description is that it can equally be applied to representations which are not necessarily highest weight representations.

The important insight of Nahm in [11] was that

$$(\mathcal{H}_1 \otimes \mathcal{H}_2)_f^0 \subset \mathcal{H}_1^s \otimes \mathcal{H}_2^0 \quad \text{and} \quad (\mathcal{H}_1 \otimes \mathcal{H}_2)_f^0 \subset \mathcal{H}_1^0 \otimes \mathcal{H}_2^s. \quad (2.11)$$

This was shown by giving an algorithm for reducing the left-hand-side, using the two comultiplications. We want to conjecture here (and give a proof for the case of the Virasoro algebra), that actually much more is true, namely

$$(\mathcal{H}_1 \otimes \mathcal{H}_2)_f^n \subset \mathcal{H}_1^s \otimes \mathcal{H}_2^n \quad \text{and} \quad (\mathcal{H}_1 \otimes \mathcal{H}_2)_f^n \subset \mathcal{H}_1^n \otimes \mathcal{H}_2^s. \quad (2.12)$$

The strategy to show this is again to give an algorithm for reducing the left-hand-side. We have been unable so far to show that the algorithm terminates in general. However, for the Virasoro algebra which we shall study for most of the rest of the paper it is easy to see that it does.

The importance of this result is that it enables one to analyse the fusion product level by level, in a way which respects the action of the modes. For example, the comultiplication formula gives an action of S_m ,

$$S_m : (\mathcal{H}_1 \otimes \mathcal{H}_2)_f^n \rightarrow (\mathcal{H}_1 \otimes \mathcal{H}_2)_f^{n-m} \quad (2.13)$$

for $m < n$. In particular, the zero modes map each space into itself, and we can thus analyse the eigenvalues and eigenvectors up to any level.

For the application we have primarily in mind — the analysis of not completely reducible representations of the Virasoro algebra — it is essential to be able to analyse all of these spaces. In particular as we shall see, there exist not completely reducible representations which seem to be completely reducible if only the lowest energy space of the fusion product is analysed in the way proposed by Nahm [11]. The method used here not only shows that

they are in fact not completely reducible, but it also allows one to calculate certain *a priori* free parameters which characterise different representations of the same general type. More fundamentally, (2.12) gives an upper bound on the number of states of a certain level in the fusion product. In particular, it reaffirms the interpretation of Nahm about the dimension of the special subspace being closely related to the quantum dimension of the corresponding representation. It should also have implications for questions of convergence of characters.

As mentioned before, the strategy for proving (2.12) is to give an algorithm for reducing the left-hand-side. We want to explain this algorithm now in more detail. To fix notation let us assume (for simplicity) that the two parameters are $(z_1, z_2) = (1, 0)$. We shall only consider the proof for the first case, the other case being completely analogous because of symmetry. The algorithm consists essentially of two steps which are applied alternately.

(A1): Rewrite a given vector

$$\psi_1 \otimes \psi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2 \quad (2.14)$$

as

$$\psi_1 \otimes \psi_2 = \sum_i \psi_1^i \otimes \psi_2^i + \Delta_{1,0}(\mathcal{A}_{n+1}) (\mathcal{H}_1 \otimes \mathcal{H}_2) , \quad (2.15)$$

where $\psi_1^i \in \mathcal{H}_1^s$.

(A2): Rewrite $\psi_1 \otimes \psi_2 \in \mathcal{H}_1^s \otimes \mathcal{H}_2$ as

$$\psi_1 \otimes \psi_2 = \sum_i \psi_1^i \otimes \psi_2^i + \Delta_{1,0}(\mathcal{A}_{n+1}) (\mathcal{H}_1 \otimes \mathcal{H}_2) , \quad (2.16)$$

where $\psi_1^i \in \mathcal{A}_-^0 \mathcal{H}_1^s$ and $\psi_2^i \in \mathcal{H}_2^n$

The first step **(A1)** requires some comment. We first rewrite $\psi_1 \in \mathcal{H}_1$ as

$$\psi_1 = \sum_j \psi_j^s + \sum_k \mathcal{A}_{--} \chi_k^s , \quad (2.17)$$

where ψ_j^s and χ_k^s are in \mathcal{H}_1^s . To rewrite the action of \mathcal{A}_{--} on the left-hand vector in the fusion tensor product, we use the following crucial property which follows from [10, 14],

$$\begin{aligned} \tilde{\Delta}_{0,-1}(S_{-m}) &= (e^{L-1} \otimes e^{L-1}) \Delta_{1,0}(S_{-m}) (e^{-L-1} \otimes e^{-L-1}) \\ &= \Delta_{1,0}(e^{L-1} S_{-m} e^{-L-1}) \\ &= \sum_{l=m}^{\infty} \binom{l-h}{m-h} \Delta_{1,0}(S_{-l}) \\ &= \sum_{l=m}^n \binom{l-h}{m-h} \Delta_{1,0}(S_{-l}) + \Delta_{1,0}(\mathcal{A}_{n+1}) , \end{aligned} \quad (2.18)$$

where h is the conformal weight of S , and we have assumed that $m \leq n$, as otherwise the whole expression is in the subspace by which we quotient. We note that for $m \leq l \leq n$, $\Delta_{1,0}(S_{-l})$ is of the form $\mathcal{A}_{-}^0 \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{A}_{-}$. Furthermore, we have

$$\tilde{\Delta}_{0,-1}(S_{-m}) = (S_{-m} \otimes \mathbb{1}) + \varepsilon_1 \sum_{l=1-h}^{\infty} \binom{m+l-1}{m-h} (-1)^{l+h-1} (\mathbb{1} \otimes S_l), \quad (2.19)$$

and we can thus use these equations to rewrite $(S_{-m} \otimes \mathbb{1})$ in terms of \mathcal{A}_{-}^0 acting on the left hand vector (and some modes acting on \mathcal{H}_2). We then rewrite these terms again as vectors in \mathcal{H}_1^s plus terms of the form $\mathcal{A}_{-}\phi_l^s$, where $\phi_l^s \in \mathcal{H}_1^s$ and repeat the procedure. Using the same argument as in [11], it is easy to see that this algorithm always stops, as the conformal weight of the relevant vectors decreases in each step.

To understand how the second step **(A2)** can be implemented, we note that for a monomial of negative modes S_{-I} of level $|I|$, the comultiplication formulae (2.2) and (2.3) imply that

$$\Delta_{1,0}(S_{-I}) = (\mathbb{1} \otimes S_{-I}) + \sum_k \mathcal{A}_{-}^0 \otimes P_{-I_k}^{(k)}, \quad (2.20)$$

where $P_{-I_k}^{(k)}$ are monomials with $|I_k| < |I|$. Using this relation repeatedly, it is clear that **(A2)** can be implemented.

We want to explain now how **(A1)** and **(A2)** can be used to reduce the left hand side of (2.12). We first apply **(A1)** to replace all terms in the left hand vector which are not in the special subspace. We then use **(A2)** to remove all states of level higher than n from the right hand vector, in the course of which we potentially create left hand vectors which are outside the special subspace. We thus repeat step **(A1)**, thereby obtaining terms where arbitrary modes up to (negative) level n act on the right hand vectors. Thus, potentially we have to repeat step **(A2)**, and so on.

In order to show that the algorithm stops one could try to find a real-valued function of the two vectors which decreases in each cycle. Unfortunately, it is not clear how to define such a function in general as the specific form of the null-vector (which is used to rewrite a vector in $\mathcal{A}_{-}^0 \psi^0$, where ψ^0 is the highest weight vector, as an element in the special subspace and vectors of the form $\mathcal{A}_{-}\psi$) is crucial in step **(A1)**. We do not know general properties of these null-vectors and thus have not been able to show the result in general. On the other hand for the case of the Virasoro algebra, we can define the decreasing function to be the sum of the number of modes of L on both vectors. It is clear that this function does not increase in step **(A2)**, and in step **(A1)** it has to decrease when we replace an element in $L_{-1}^n \psi^0$ by vectors involving L_{-2} or higher modes. A similar construction also works for certain representations of the W_3 algebra.

3 Virasoro fusion revisited

Let us now specifically consider the Virasoro algebra Vir which is generated by the modes $L_n, n \in \mathbb{Z}$ of the stress tensor with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}. \quad (3.1)$$

The representations of this algebra which are most relevant in conformal field theory are *highest weight representations* of definite highest weight h ; these are generated by the creation modes $L_n, n < 0$ from a highest weight vector v_h , satisfying

$$\begin{aligned} L_0 v_h &= h v_h, \\ L_n v_h &= 0, \quad \text{for } n > 0. \end{aligned} \quad (3.2)$$

Here we assume in particular that the highest weight vector is a cyclic vector with respect to the action of the Virasoro algebra, *i.e.* that all vectors in the representation space can be obtained by the action of the Virasoro algebra from the highest weight vector. We shall encounter later on representations which contain a vector of lowest L_0 -eigenvalue, satisfying (3.2), but for which this vector is not cyclic. These representations shall be called *generalised highest weight representations*.

We parametrise the central charge of the Virasoro algebra as

$$c = 13 - 6(t + t^{-1}), \quad (3.3)$$

and the conformal weights of highest weight vectors

$$h = \frac{\alpha^2}{4t} - \frac{(t-1)^2}{4t}. \quad (3.4)$$

The highest weight representation \mathcal{M}_h which is freely generated from v_h by the action of

$$\text{Vir}_- := \langle L_n | n < 0 \rangle \quad (3.5)$$

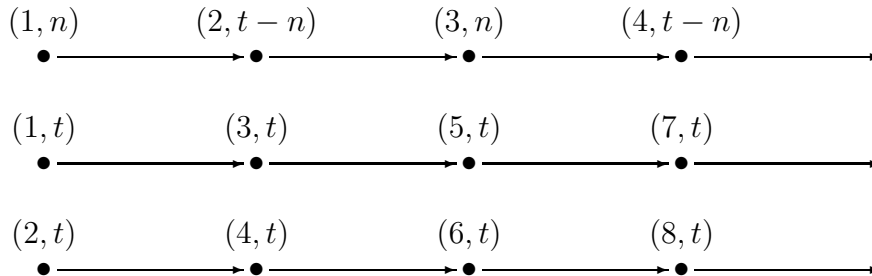
is called a Verma module. It may contain a non-zero highest weight vector w of weight $h + n$, and such a vector is then called a *singular vector* of degree n . It generates a subrepresentation of \mathcal{M}_h , and conversely all subrepresentations of \mathcal{M}_h are generated from highest weight vectors.

The Verma module \mathcal{M}_h has a singular vector of degree $N = mn$ if $h = h(\alpha_{m,n})$, where $\alpha_{m,n} = mt - n$ and m and n are positive integers. In this case we will denote the corresponding Verma module by $\mathcal{M}_{m,n}$.

If t is an integer all singular vectors of \mathcal{M}_h are of this type. Furthermore, as $\alpha_{m,n} = \alpha_{m+1,n+t}$ and $h(\alpha_{m,n}) = h(\alpha_{-m,-n})$, we can always reduce the label n to lie in the *fundamental domain*, $0 < n \leq t$. Then

$$h_{m,n} + mn = h_{m,-n} = h_{m+1,t-n}$$

and thus the subrepresentation generated by the singular vector possesses again a singular vector. For integer t we hence obtain the following chains of Verma module embeddings:



Here, a vertex with a label (m, n) denotes a highest weight vector of weight $h_{m,n}$, and an arrow $1 \longrightarrow 2$ means that the vertex 2 is the singular vector of lowest degree contained in the representation generated by 1.

In the following we shall also need the Verma module embedding diagrams for irrational t . In this case there are two types of diagrams: those corresponding to Verma modules where the highest weight vector is described by $m, n \in \mathbb{N}$ and where there exists precisely one singular vector of weight $h_{m,-n}$ (at level mn), and those for which this is not the case, and which do not have any singular vectors at all.

Any highest weight representation can be obtained as the quotient of a Verma module by a subrepresentation. The singular vectors which generate the corresponding subrepresentation are then called *null vectors*. For integer $m, n \in \mathbb{N}$, we denote by $\mathcal{V}_{m,n}$ the highest weight representation of weight $h_{m,n}$ which is obtained from the Verma module $\mathcal{M}_{m,n}$ by dividing out the subrepresentation generated from the singular vector at level mn (which is then null). For irrational t , all such representations are irreducible, and for integer t , precisely those are irreducible for which n is in the fundamental domain $0 < n \leq t$. (For $n > t$, the Verma module $\mathcal{M}_{m,n}$ is the same as $\mathcal{M}_{m',n'}$, where n' is in the fundamental domain; $\mathcal{M}_{m,n}$ therefore possesses a singular vector of level $m'n' < mn$, and $\mathcal{V}_{m,n}$ is not irreducible.)

All these representations are quasirational, and hence their fusion product decomposes into finitely many indecomposable quasirational subrepresentations. Using the techniques of the previous section we can analyse their fusion, and we shall do so later on in some detail. First, however, we want to see, what can be learned about these fusion products using the general results of Feigin and Fuchs [16, 17]. From their point of view, the fusion of $\phi_1 \in \mathcal{V}_1$ and $\phi_2 \in \mathcal{V}_2$ is the set of all representations \mathcal{P}_3 , for which there exists $\phi_3 \in \mathcal{P}_3$, such that the three-point function $\langle \phi_3(z_3) \phi_2(z_2) \phi_1(z_1) \rangle$ does not vanish. (Here $\phi(z)$ denotes the field corresponding to the state $\phi \in \mathcal{P}$.) Using the Möbius invariance of the three-point functions, it is sufficient to consider the case, where the three points z_1, z_2, z_3 are $0, 1, \infty$. We can furthermore identify fields and states at $z = 0$.

Suppose now that $\phi_i \equiv \phi_{m_i, n_i} \in \mathcal{V}_{m_i, n_i}$ are primary fields, and that $m_1, m_2, n_1, n_2 \in \mathbb{N}$. We want to use the fact that the three-point function has to vanish if we consider the null descendent \mathcal{N}_1 of ϕ_1 at level $m_1 n_1$. This gives the equation

$$0 = \langle \phi_3(\infty) \phi_2(1) \mathcal{N}_1(0) \rangle = p_{m_1, n_1}(h_2, h_3) \langle \phi_3(\infty) \phi_2(1) \phi_1(0) \rangle, \quad (3.6)$$

where p_{m_1, n_1} factorises as [17]

$$p_{m_1, n_1}(h_2, h_3) \propto \prod_{r,s} (\alpha_2 + \alpha_3 - \alpha_{r,s})(\alpha_2 - \alpha_3 - \alpha_{r,s}). \quad (3.7)$$

Here we use the parametrisation (3.4), and the product is over the range

$$\begin{aligned} r &\in \{-(m_1 - 1), -(m_1 - 3), \dots, (m_1 - 1)\}, \\ s &\in \{-(n_1 - 1), -(n_1 - 3), \dots, (n_1 - 1)\}. \end{aligned} \quad (3.8)$$

We can also use the null-vector relations for ϕ_2 , to obtain a formula similar to (3.6). In order for the fusion of ϕ_1 and ϕ_2 to contain the highest weight representation generated

from ϕ_3 , the three-point function of some states in the three representations has to be non-trivial. It is easy to see that this can only be the case if the three-point function of the corresponding highest weight vectors does not vanish, and this gives rise, by (3.6), to the (unintersected) fusion rules

$$\phi_{m_1, n_1} \times \phi_{m_2, n_2} = \sum_{m_3=|m_1-m_2|+1}^{m_1+m_2-1} \sum_{n_3=|n_1-n_2|+1}^{n_1+n_2-1} \phi_{m_3, n_3}, \quad (3.9)$$

where the sums are over every other integer. In particular, this implies that ϕ_3 has to generate a degenerate representation, *i.e.* that the Verma module generated from ϕ_3 has a singular vector. Because of the invariance of the three-point functions under Möbius transformations and the symmetry of the fusion rules (3.9) under cyclic permutations, it is then clear that the three-point function will also satisfy the condition coming from the null-vector of ϕ_{m_3, n_3} at level $m_3 n_3$.

In general however, this is not sufficient to guarantee that the *irreducible* representation generated from ϕ_3 possesses a non-vanishing three-point function with ϕ_1 and ϕ_2 — for example for integer t , if n_3 is not in the fundamental domain, ϕ_{m_3, n_3} has typically another singular vector at lower level, which might then not be null in the three-point function. On the other hand, it is clear that (3.9) gives the correct fusion rules for the corresponding irreducible representations if t is irrational, as the degenerate representations possess only one singular vector in this case. In the notation of the previous section this means

$$(\mathcal{V}_{m_1, n_1} \otimes \mathcal{V}_{m_2, n_2})_f = \bigoplus_{m_3=|m_1-m_2|+1}^{m_1+m_2-1} \bigoplus_{n_3=|n_1-n_2|+1}^{n_1+n_2-1} \mathcal{V}_{m_3, n_3}. \quad (t \text{ irrational}) \quad (3.10)$$

Before turning to the case of integer t , let us mention that this result gives also rise to a character identity. It is clear from the previous section that we can define a character for the fusion tensor product by setting

$$\chi_{(\mathcal{V}_{m_1, n_1} \otimes \mathcal{V}_{m_2, n_2})_f}(\tau) = \lim_{N \rightarrow \infty} \text{Tr} \Big|_{(\mathcal{V}_{m_1, n_1} \otimes \mathcal{V}_{m_2, n_2})_f^N} \left(q^{L_0 - c/24} \right), \quad (3.11)$$

where $q = \exp(2\pi i \tau)$. For irrational t we then obtain

$$\chi_{(\mathcal{V}_{m_1, n_1} \otimes \mathcal{V}_{m_2, n_2})_f}(\tau) = \sum_{m_3=|m_1-m_2|+1}^{m_1+m_2-1} \sum_{n_3=|n_1-n_2|+1}^{n_1+n_2-1} \chi_{m_3, n_3}(\tau), \quad (3.12)$$

where the character $\chi_{m, n}(\tau)$ is the character of $\mathcal{V}_{m, n}$ which is explicitly given as

$$\chi_{m, n}(\tau) = \text{Tr}_{\mathcal{V}_{m, n}} q^{L_0 - c/24} = \eta(\tau)^{-1} q^{\frac{1-c}{24}} \left(q^{h_{m, n}} - q^{h_{m, -n}} \right), \quad (3.13)$$

and $\eta(\tau)$ is the Dedekind η function

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n). \quad (3.14)$$

Let us now turn to analysing the case where t is an integer. In order to be more specific, let us denote by \mathcal{W}_3 the subspace of \mathcal{M}_{m_3, n_3} which is null in the three-point function with ϕ_1 and ϕ_2 , and all their descendants. Because of the invariance of the three-point function with respect to the action of the Virasoro algebra, this subspace has to be a subrepresentation and therefore is generated from a singular vector in \mathcal{M}_{m_3, n_3} . If we are in the fundamental domain ($n_3 \leq t$), then \mathcal{W}_3 is generated from the singular vector at level $m_3 n_3$, and the quotient space $\mathcal{M}_{m_3, n_3} / \mathcal{W}_3$ is irreducible. On the other hand, this is not always the case, even if we start with irreducible representations ($n_1, n_2 \leq t$). Indeed, if $t < n_3 \leq 2t$, then the embedding diagram of the representation corresponding to ϕ_{m_3, n_3} is

$$\begin{array}{l} \mathcal{V}_{1, n_3}: \quad \begin{array}{ccccc} (1, 2t - n_3) & & (2, n_3 - t) & & (3, 2t - n_3) \\ \bullet & \xrightarrow{\quad} & \bullet & \text{---} & \times \end{array} \\ \\ \mathcal{V}_{m_3, n_3}: \quad \begin{array}{ccccccc} (m_3 - 1, n_3 - t) & & (m_3, 2t - n_3) & & (m_3 + 1, n_3 - t) & & (m_3 + 2, 2t - n_3) \\ \bullet & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet & \text{---} & \times \end{array} \end{array}$$

where in the second line $m_3 \geq 2$, and \times denotes the fundamental null vector (corresponding to the singular vector at level $m_3 n_3$) which certainly generates a subspace contained in \mathcal{W}_3 . The interesting question is then whether \mathcal{W}_3 is actually larger, and whether it contains also the subrepresentations generated from the singular vectors $(2, n_3 - t)$, $(m_3, 2t - n_3)$ and $(m_3 + 1, n_3 - t)$.

The above embedding diagram implies that the null vector condition for ϕ_{m_3, n_3} and $t < n_3 < 2t$ factorises as

$$\begin{aligned} 0 = p_{1, n_3}(\alpha_1, \alpha_2) &= p_{1, 2t - n_3}(\alpha_1, \alpha_2) p_{2, n_3 - t}(\alpha_1, \alpha_2), \\ 0 = p_{m_3, n_3}(\alpha_1, \alpha_2) &= p_{m_3 - 1, n_3 - t}(\alpha_1, \alpha_2) p_{m_3, 2t - n_3}(\alpha_1, \alpha_2) p_{m_3 + 1, n_3 - t}(\alpha_1, \alpha_2). \end{aligned}$$

Furthermore, $t < n_3 \leq n_1 + n_2 - 1$ implies that $2t - n_3 \geq 2t - n_1 - n_2 + 1 \geq |n_1 - n_2| + 1$; hence, if the right hand side of the fusion rule (3.9) contains a field ϕ_{m_3, n_3} , it also contains the field $\phi_{m_3, 2t - n_3}$. This implies that $p_{m_3, 2t - n_3}(\alpha_1, \alpha_2) = 0$, and thus

$$\begin{aligned} 0 &= p_{1, 2t - n_3}(\alpha_1, \alpha_2), \\ 0 &= p_{m_3 - 1, n_3 - t}(\alpha_1, \alpha_2) p_{m_3, 2t - n_3}(\alpha_1, \alpha_2) = p_{1, m_3 t + t - n_3}(\alpha_1, \alpha_2). \end{aligned}$$

It follows that the singular vectors $(2, n_3 - t)$ and $(m_3 + 1, n_3 - t)$ are indeed in \mathcal{W}_3 . On the other hand, the singular vector $(m_3, 2t - n_3)$ is not in \mathcal{W}_3 , since the field $\phi_{m_3 - 1, n_3 - t}$ (whose conformal weight equals that of ϕ_{m_3, n_3}) is not in the fusion product because of parity, and thus

$$p_{m_3 - 1, n_3 - t}(\alpha_1, \alpha_2) \neq 0.$$

Hence we have

$$(\mathcal{M}_{m_3, n_3} / \mathcal{W}_3) = \begin{cases} \mathcal{V}_{m_3, n_3} & \text{if } n_3 \leq t, \\ \mathcal{V}_{1, m_3 t + t - n_3} & \text{if } n_3 > t. \end{cases} \quad (3.15)$$

We have thus shown, that the fusion product has non-vanishing correlation functions with these reducible subrepresentations. This is not yet sufficient to determine the fusion product uniquely, as we can (in this way) only study correlation functions of the fusion product with highest weight representations (which have in particular a cyclic highest weight vector). However, as will become apparent from the explicit calculations, the fusion product typically contains generalised highest weight representations for which this is not true. On the other hand, the above analysis certainly imposes constraints on the possible structure of the fusion product.

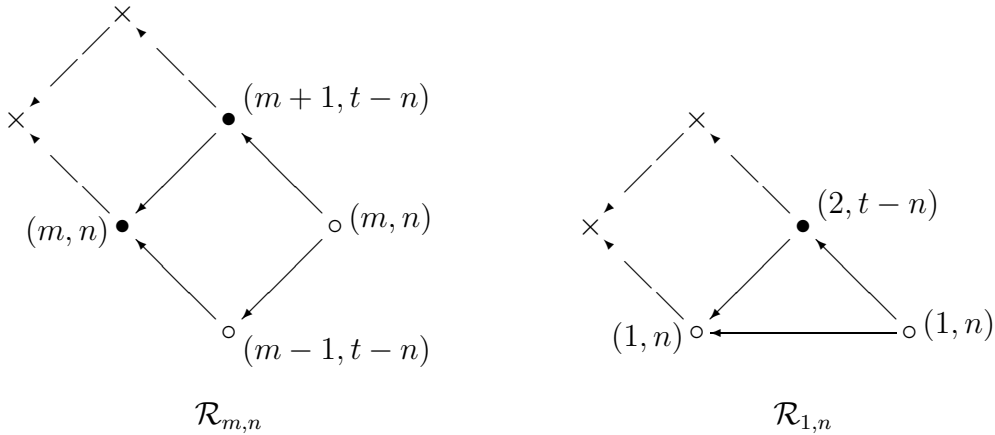
4 The structure of the fusion product

We shall now explain what we conjecture to be the structure of the fusion product. We shall then check that it satisfies the constraints of the above analysis. In the next section we shall present the results of the analysis of the fusion product using the algorithm of Section 2, which confirm these conjectures.

As is clear from the above arguments, the highest weight of the representation $\mathcal{M}_{m,n}$ with $2t > n > t$ differs from that of the representation $\mathcal{M}_{m,2t-n}$ by an integer. It is then possible that the two representations couple to form a generalised highest weight representation. Indeed, this is what seems to happen, and we conjecture that the decomposition of the fusion product for integer t is

$$(\mathcal{V}_{m_1, n_1} \otimes \mathcal{V}_{m_2, n_2})_f = \begin{cases} \bigoplus_m \bigoplus_{n=|n_1-n_2|+1}^{n_1+n_2-1} \mathcal{V}_{m,n} & \text{if } n_1 + n_2 \leq t, \\ \bigoplus_m \left[\left(\bigoplus_{n=|n_1-n_2|+1}^{2t-n_1-n_2-1} \mathcal{V}_{m,n} \right) \oplus \left(\bigoplus_{n=2t-n_1-n_2+1}^{t-1} \mathcal{R}_{m,n} \right) \oplus \underline{\mathcal{V}_{m,t}} \right] & \text{if } n_1 + n_2 > t, \end{cases} \quad (4.1)$$

where we sum over $m = |m_1 - m_2| + 1, \dots, m_1 + m_2 - 3, m_1 + m_2 - 1$ and $n + n_1 + n_2$ odd. The underlined term is only present if $t + n_1 + n_2$ is odd. $\mathcal{R}_{m,n}$ denotes the decomposable representation which can schematically be represented by



Here, each vertex \bullet or \circ with label (r, s) denotes a state of conformal weight $h_{r,s}$ and the vertices \times correspond to null vectors. An arrow $A \longrightarrow B$ indicates that the vertex B is in the image of A under the action of the Virasoro algebra. Conformal weights are constant horizontally, and increase vertically. The vertices \circ denote states which are not descendents, *i.e.* states which cannot be obtained by the action of the negative modes from other states.

We should mention that the indecomposable representations $\mathcal{R}_{m,n}$ with $m \geq 2$ appear to be decomposable if only the lowest energy space of the product is analysed in the way proposed by Nahm [11]; to see that they are in fact indecomposable it is necessary to analyse the product space up to level $(m-1)(t-n)$.

Before describing these representations in more detail, let us first explain, why this is consistent with the above analysis. Because of Möbius covariance we can consider the case where the highest weight representation is inserted at infinity. Then, by a standard argument of conformal field theory, it is clear that any singular vector in the highest weight representation at infinity can only get a non-vanishing contribution from states in the fusion product which are not descendents; these states have been denoted by \circ in the above diagram.

Let us now consider the three point function $\langle \phi_3 \phi_2 \phi_1 \rangle$, and in particular the contribution of the subrepresentation $\mathcal{R}_{m,n}$ in the fusion product of ϕ_1 and ϕ_2 . For $m \geq 2$, the first singular vector $(m_3 + 1, t - n_3)$ of the highest weight representation $(m_3, n_3) = (m, n)$ (where $0 < n < t$) is null in the correlation function with the representation $\mathcal{R}_{m,n}$, as there does not exist a non-descendent state in $\mathcal{R}_{m,n}$ at this level. The same is also true for the second singular vector $(m_3 + 1, n_3 - t) = (m + 1, t - n)$ of the highest weight representation $(m_3, n_3) = (m, 2t - n)$. On the other hand, the first singular vector $(m_3, 2t - n_3) = (m, n)$ of the highest weight representation $(m_3, n_3) = (m, 2t - n)$ is not null (with respect to $\mathcal{R}_{m,n}$), as it gets a non-vanishing contribution from the \circ -state (m, n) in $\mathcal{R}_{m,n}$.

For $m = m_3 = 1$, all singular vectors of the two highest weight representations at infinity vanish in correlation functions with $\mathcal{R}_{m,n}$, as all states in $\mathcal{R}_{m,n}$ of the corresponding level are descendents. Thus the conjecture is consistent with the above findings.

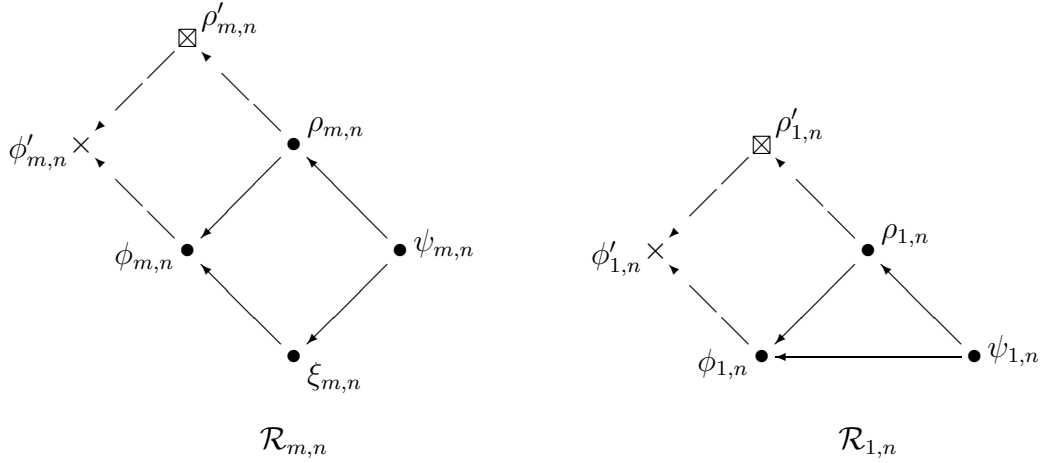
The conjecture also satisfies another constraint which comes from the analysis of the characters. We can easily read off the character of $\mathcal{R}_{m,n}$ from the diagram, and find

$$\chi_{\mathcal{R}_{m,n}}(\tau) = \chi_{\mathcal{V}_{m,n}}(\tau) + \chi_{\mathcal{V}_{m,2t-n}}(\tau) \quad (4.2)$$

$$= \begin{cases} 2\chi_{1,n}(\tau) + \chi_{2,t-n}(\tau) & \text{for } m = 1 \\ \chi_{m-1,t-n}(\tau) + 2\chi_{m,n}(\tau) + \chi_{m+1,t-n}(\tau) & \text{for } m > 1 \end{cases} \quad (4.3)$$

In particular, this implies that the character of the fusion product satisfies (3.12), even for integer t . This is in accordance with the observation that the structure of the character (3.11) should be unchanged when irrational t approaches an integer, as the structure of the representations \mathcal{V}_{m_1,n_1} and \mathcal{V}_{m_2,n_2} is unchanged in this limit: the additional vectors which become singular as t becomes an integer are already in the subspace generated by the null vector at level $m_1 n_1$ and $m_2 n_2$, respectively.

We close this section by describing the structure of the generalised highest weight representations $\mathcal{R}_{m,n}$ in more detail. In the following diagram we have denoted actual states of $\mathcal{R}_{m,n}$ by \bullet , and vectors in the Verma module which are null in $\mathcal{R}_{m,n}$ by \times if they are singular, and by \boxtimes if they are sub-singular. (The distinction between singular and sub-singular will be explained later.)



The representation $\mathcal{R}_{m,n}$ contains a subrepresentation $\mathcal{V}_{1,(m-1)t+n}$ which is generated from a highest weight vector $\xi_{m,n}$ of weight $h_{m-1,t-n}$, *i.e.*

$$\begin{aligned} L_0 \xi_{m,n} &= h_{m-1,t-n} \xi_{m,n} , \\ L_p \xi_{m,n} &= 0 , \quad \text{for } p \geq 1 . \end{aligned}$$

Unless $m = 1$, the representation $\mathcal{V}_{1,(m-1)t+n}$ is reducible and we denote by $\phi_{m,n}$ its lowest singular vector,

$$\sigma_{m-1,t-n} \xi_{m,n} = \phi_{m,n} .$$

Here $\sigma_{p,q}$ denotes the monomial of negative Virasoro modes which generates the singular vector at level pq from the highest weight vector; we have fixed the normalisation of $\sigma_{p,q}$ by requiring that the coefficient of L_{-1}^{pq} is 1.

The highest weight representation $\mathcal{V}_{m,n}$ generated from $\phi_{m,n}$ is an irreducible subrepresentation of $\mathcal{R}_{m,n}$,

$$\begin{aligned} L_0 \phi_{m,n} &= h_{m,n} \phi_{m,n} , \\ \sigma_{m,n} \phi_{m,n} &= \phi'_{m,n} \equiv 0 . \end{aligned}$$

The total representation $\mathcal{R}_{m,n}$ is generated from a cyclic vector $\psi_{m,n}$ which is not an eigenvector of L_0 but forms a Jordan cell of dimension two with the vector $\phi_{m,n}$ of conformal weight $h_{m,n}$,

$$\begin{aligned} L_0 \psi_{m,n} &= h_{m,n} \psi_{m,n} + \phi_{m,n} , \\ L_0 \phi_{m,n} &= h_{m,n} \phi_{m,n} . \end{aligned}$$

Unless $m = 1$, $\psi_{m,n}$ is not a highest weight vector, but positive Virasoro modes map $\psi_{m,n}$ to descendants of $\xi_{m,n}$. We can redefine $\psi_{m,n}$ (by adding descendants of $\xi_{m,n}$) so that it is annihilated by L_p with $p \geq 2$. (This is possible because there is only one singular vector in the space of descendants of $\xi_{m,n}$ at level $(m-1)(t-n)$.) This fixes the freedom to define $\psi_{m,n}$ up to the addition of multiples of $\phi_{m,n}$. $L_1\psi_{m,n}$ (which does not depend on this freedom) is then annihilated by L_p with $p \geq 2$, and since there is only one such state at level $(m-1)(t-n) - 1$ in the representation generated from $\xi_{m,n}$ (as the inner product is non-degenerate), it is unique up to a factor.⁵ It follows that $L_1^{(m-1)(t-n)}\psi_{m,n}$ does not vanish (if $L_1\psi_{m,n} \neq 0$), and we can thus describe the remaining factor by

$$\begin{aligned} L_1^{(m-1)(t-n)}\psi_{m,n} &= \beta_{m,n} \xi_{m,n} \\ L_p\psi_{m,n} &= 0, \quad \text{for } p \geq 2. \end{aligned} \quad (4.4)$$

Different values for $\beta_{m,n}$ give rise to inequivalent representations. In particular, we cannot scale $\beta_{m,n}$ away by redefining $\psi_{m,n}$ or $\xi_{m,n}$, as we have the additional relation

$$(L_0 - h_{m,n})\psi_{m,n} = \sigma_{m-1,t-n}\xi_{m,n}. \quad (4.5)$$

For the specific representations obtained in the fusion product of known representations these parameters are determined. We have calculated $\beta_{m,n}$ in some cases, and have included the explicit results in the next section.

The Verma module corresponding to $\mathcal{R}_{m,n}$, *i.e.* the module which is generated by the action of the negative modes on $\xi_{m,n}$ and $\psi_{m,n}$, contains a sub-singular vector $\rho_{m,n}$ of weight $h_{m+1,t-n}$. This means that $\rho_{m,n}$ is not singular in the Verma module itself, but that it becomes singular when we quotient the Verma module by the subrepresentation generated from $\phi_{m,n}$. (Another way of saying this is that the positive modes map $\rho_{m,n}$ to descendants of $\phi_{m,n}$.) $\rho_{m,n}$ is not null in $\mathcal{R}_{m,n}$, since the singular vector $\phi_{m,n}$ is not null either. We normalise $\rho_{m,n}$ by requiring that it is of the form

$$\rho_{m,n} = \sigma_{m,n}\psi_{m,n} + L_{-P}\xi_{m,n}, \quad (4.6)$$

where $|P| = (m-1)t + n$, and $\sigma_{p,q}$ is as before.

The next sub-singular vector in the Verma module is of weight $h_{m+2,n}$ and is denoted by $\rho'_{m,n}$. It is actually null in $\mathcal{R}_{m,n}$, as the positive modes map $\rho'_{m,n}$ to descendants of $\phi'_{m,n}$ which is also null. Again, we can assume that $\rho'_{m,n}$ is of the form

$$\rho'_{m,n} = \sigma_{m+1,t-n}\rho_{m,n} + L_{-Q}\phi_{m,n}, \quad (4.7)$$

where $|Q| = (m+1)t - n$.

For $m = 1$, the situation is degenerate, as $\xi_{1,n} = 0$, and $\psi_{1,n}$ is annihilated by all positive modes,

$$L_n\psi_{m,n} = 0, \quad \text{for } n > 0. \quad (4.8)$$

⁵We thank Falk Rohsiepe for pointing out to us that there is only one characteristic parameter in this case. See also [12] for a more detailed analysis of these representations.

In particular, this means that all representations of this type are equivalent, and that in this case there is no free parameter labelling inequivalent representations.

We can also explicitly give a basis for $\mathcal{R}_{m,n}^N$, the subspace of $\mathcal{R}_{m,n}$ of all descendents up to level N ; it can be taken to be the following subset of the lexicographically ordered states

$$\begin{aligned}
& \cdots L_{-2}^{k_2} L_{-1}^{k_1} \xi_{m,n}, \quad \sum_j j k_j \leq N, \quad k_1 < (m-1)(t-n), \\
& \cdots L_{-2}^{k'_2} L_{-1}^{k'_1} \phi_{m,n}, \quad \sum_j j k_j \leq N - (m-1)(t-n), \quad k_1 < mn, \\
& \cdots L_{-2}^{k'_2} L_{-1}^{k'_1} \psi_{m,n}, \quad \sum_j j k_j \leq N, \quad k_1 < mn, \\
& \cdots L_{-2}^{k''_2} L_{-1}^{k''_1} \rho_{m,n}, \quad \sum_j j k_j \leq N - mn, \quad k_1 < (m+1)(t-n).
\end{aligned} \tag{4.9}$$

5 Explicit Calculations

In this section we describe the explicit calculations which we have done in order to determine the fusion product of some of the representations. We shall outline first how we proceeded in general, before giving our results in detail. For one example we have included the explicit matrices, describing the action of the positive Virasoro generators on the fusion product, in the appendix.

Let us suppose that \mathcal{V}_1 and \mathcal{V}_2 are two quasirational representations, and that we want to analyse their fusion product up to a given level L .⁶ We consider first the space $\mathcal{F}_{12} = \mathcal{V}_1^s \otimes \mathcal{V}_2^L$, which contains the space we are interested in, $(\mathcal{V}_1 \otimes \mathcal{V}_2)_f^L$, by the arguments of Section 2. Typically \mathcal{F}_{12} is too large, reflecting that there exists a ‘spurious subspace’ [11]. To find the additional relations we calculate $\Delta(a)\Psi$, where $\Psi \in \mathcal{F}_{12}$ and $a \in \mathcal{A}_{L+1}$, using the comultiplication. We then use the null-relations in \mathcal{V}_1 and \mathcal{V}_2 to rewrite the result, and apply then the reduction algorithm of Section 2 to obtain an expression in \mathcal{F}_{12} . By construction we know that this expression is in the subspace by which we have to quotient $(\mathcal{V}_1 \otimes \mathcal{V}_2)_f$ in order to obtain $(\mathcal{V}_1 \otimes \mathcal{V}_2)_f^L$; it may, however, happen that the resulting expression is not identically zero, and this gives then rise to a relation in \mathcal{F}_{12} .

A priori, it is not clear how to find all missing relations. We took a to be all monomials of a fixed weight greater than L , and $\Psi \in \mathcal{V}_1^s \otimes \mathcal{V}_2^0$. In most cases it was sufficient to consider monomials of weight $L+1$ but in some cases we had to increase successively the weight of a up to $L+5$ to reduce the space \mathcal{F}_{12} to the dimension conjectured in the previous section. We also checked (in some cases) that no further relations arose when we did the calculation for some monomials a of higher weight.

Having obtained the space $(\mathcal{V}_1 \otimes \mathcal{V}_2)_f^L$ we construct the spaces $(\mathcal{V}_1 \otimes \mathcal{V}_2)_f^n$ for $0 \leq n < L$ by successively imposing the constraints $\Delta(a) = 0$ for all a of weight $n+1$. The Virasoro generators map

$$L_m: (\mathcal{V}_1 \otimes \mathcal{V}_2)_f^n \rightarrow (\mathcal{V}_1 \otimes \mathcal{V}_2)_f^{n-m},$$

⁶For most of the following we chose $L = 6$.

and we can thus calculate the matrices, representing L_m in some basis. At first, these bases are chosen at random by the computer program, and in order to interpret the results more easily, we change to a basis in which the action is in block-diagonal form. It is then easy to check the conjectures about the decomposition of the fusion product. To read off the parameter characterising the generalised highest weight representations $\mathcal{R}_{m,n}$, we change the basis in the corresponding block to the canonical basis introduced in the previous section.

5.1 $t = 2$

We first analysed the fusion product of irreducible representations. Up to level six the decomposition was as follows

$$\begin{aligned}
(\mathcal{V}_{2,1} \otimes \mathcal{V}_{2,1})_{\text{f}} &= \mathcal{V}_{1,1} \oplus \mathcal{V}_{3,1}, \\
(\mathcal{V}_{m,1} \otimes \mathcal{V}_{1,2})_{\text{f}} &= \mathcal{V}_{m,2}, \quad \text{for } m = 1, \dots, 5, \\
(\mathcal{V}_{1,2} \otimes \mathcal{V}_{m,2})_{\text{f}} &= \mathcal{R}_{m,1}, \quad \text{for } m = 1, \dots, 5, \\
(\mathcal{V}_{2,2} \otimes \mathcal{V}_{2,2})_{\text{f}} &= \mathcal{R}_{1,1} \oplus \mathcal{R}_{3,1}, \\
(\mathcal{V}_{2,2} \otimes \mathcal{V}_{3,2})_{\text{f}} &= \mathcal{R}_{2,1} \oplus \mathcal{R}_{4,1}, \\
(\mathcal{V}_{3,2} \otimes \mathcal{V}_{3,2})_{\text{f}} &= \mathcal{R}_{1,1} \oplus \mathcal{R}_{3,1} \oplus \mathcal{R}_{5,1}.
\end{aligned}$$

We should note that the fusion of $(\mathcal{V}_{1,2} \otimes \mathcal{V}_{m,2})$ provides a way of constructing the generalised highest weight representation $\mathcal{R}_{m,1}$. We have used this to read off the characteristic parameters of $\mathcal{R}_{m,1}$, and the results are contained in Table 1. For the case of $(\mathcal{V}_{2,1} \otimes \mathcal{V}_{1,2})_{\text{f}} = \mathcal{R}_{2,1}$ we have included some more details in the appendix.

(m, n)	$h_{m,n}$	$h_{m-1,t-n}$	$\beta_{m,n}$
(1, 1)	0	0	—
(2, 1)	1	0	−1
(3, 1)	3	1	−18
(4, 1)	6	3	−2700
(5, 1)	10	6	−1587600

Table 1: Representation data of $\mathcal{R}_{m,1}$ for $t = 2$.

We have checked that all representations of type $\mathcal{R}_{m,n}$ which appear in the above decompositions are indeed equivalent representations, *i.e.* have the same characteristic parameter (see Table 1). This provides a partial consistency check of our results, as the associativity and symmetry of the fusion product imply certain restrictions. For example, as $\mathcal{V}_{1,1}$ is the

identity field, we find

$$\begin{aligned}
(\mathcal{V}_{2,2} \otimes \mathcal{V}_{2,2})_{\text{f}} &= (\mathcal{V}_{2,1} \otimes \mathcal{V}_{1,2} \otimes \mathcal{V}_{2,1} \otimes \mathcal{V}_{1,2})_{\text{f}} \\
&= (\mathcal{R}_{1,1} \otimes (\mathcal{V}_{1,1} \oplus \mathcal{V}_{3,1}))_{\text{f}} \\
&= \mathcal{R}_{1,1} \oplus (\mathcal{R}_{1,1} \otimes \mathcal{V}_{3,1})_{\text{f}} .
\end{aligned}$$

We should note that the Verma module corresponding to $\mathcal{R}_{1,1}$ contains a subrepresentation of the same structure as $\mathcal{R}_{2,1}$. Indeed, we have

$$\begin{aligned}
\psi_{2,1} &\cong \rho_{1,1} = L_{-1}\psi_{1,1} \\
\phi_{2,1} &\cong \phi'_{1,1} = L_{-1}\phi_{1,1} \\
\xi_{2,1} &\cong \phi_{1,1} = \tfrac{1}{2}L_1\rho_{1,1} .
\end{aligned} \tag{5.1}$$

However, the last line implies that its characteristic parameter ($\frac{1}{2}$) is different from that of $\mathcal{R}_{2,1}$ in Table 1 (-1).

The results suggest that we have for general m_1, m_2 ,

$$(\mathcal{V}_{m_1,1} \otimes \mathcal{V}_{m_2,1})_{\text{f}} = \bigoplus_{m=|m_1-m_2|+1}^{m_1+m_2-1} \mathcal{V}_{m,1} , \tag{5.2}$$

and that for general m

$$\begin{aligned}
(\mathcal{V}_{m,1} \otimes \mathcal{V}_{1,2})_{\text{f}} &= \mathcal{V}_{m,2} , \\
(\mathcal{V}_{m,2} \otimes \mathcal{V}_{1,2})_{\text{f}} &= \mathcal{R}_{m,1} ,
\end{aligned} \tag{5.3}$$

where $\mathcal{R}_{m,1}$ is a generalised highest weight representation of the type discussed before, whose characteristic parameter is described (for $m \leq 5$) by Table 1. This confirms the conjecture of the previous section.

In a next step we analysed the fusion involving the representation $\mathcal{R}_{1,1}$,

$$\begin{aligned}
(\mathcal{V}_{1,2} \otimes \mathcal{R}_{1,1})_{\text{f}} &= \mathcal{V}_{1,2} \oplus \mathcal{V}_{1,2} \oplus \mathcal{V}_{2,2} , \\
(\mathcal{R}_{1,1} \otimes \mathcal{R}_{1,1})_{\text{f}} &= \mathcal{R}_{1,1} \oplus \mathcal{R}_{1,1} \oplus \mathcal{R}_{2,1} .
\end{aligned} \tag{5.4}$$

Assuming that (5.2) and (5.3) hold, this is then already sufficient to derive the decomposition of all fusion products, using the associativity and symmetry of the fusion product.⁷ We find that

$$\begin{aligned}
(\mathcal{V}_{m_1,1} \otimes \mathcal{V}_{m_2,2})_{\text{f}} &= (\mathcal{V}_{m_1,1} \otimes \mathcal{V}_{m_2,1} \otimes \mathcal{V}_{1,2})_{\text{f}} \\
&= \bigoplus_m (\mathcal{V}_{m,1} \otimes \mathcal{V}_{1,2})_{\text{f}} \\
&= \bigoplus_m \mathcal{V}_{m,2} ,
\end{aligned}$$

⁷In fact, the second identity of (5.4) follows already from the first, using the associativity of the fusion product and (5.3)

where (as always in the following) we sum over $m = |m_1 - m_2| + 1, |m_1 - m_2| + 3, \dots, m_1 + m_2 - 1$. Similarly

$$\begin{aligned}
(\mathcal{V}_{m_1,2} \otimes \mathcal{V}_{m_2,2})_f &= \bigoplus_m \mathcal{R}_{m,1}, \\
(\mathcal{V}_{m_1,1} \otimes \mathcal{R}_{m_2,1})_f &= \bigoplus_m \mathcal{R}_{m,1}, \\
(\mathcal{V}_{m_1,2} \otimes \mathcal{R}_{m_2,1})_f &= \bigoplus_m \left(\mathcal{V}_{m,2} \oplus \mathcal{V}_{m,2} \oplus \underline{\mathcal{V}_{m-1,2}} \oplus \mathcal{V}_{m+1,2} \right), \\
(\mathcal{R}_{m_1,1} \otimes \mathcal{R}_{m_2,1})_f &= \bigoplus_m \left(\mathcal{R}_{m,1} \oplus \mathcal{R}_{m,1} \oplus \underline{\mathcal{R}_{m-1,1}} \oplus \mathcal{R}_{m+1,1} \right),
\end{aligned} \tag{5.5}$$

where the underlined terms on the right hand side are absent for $m = 1$. This demonstrates in particular that the set of representations $\mathcal{R}_{m,1}$, $\mathcal{V}_{m,1}$ and $\mathcal{V}_{m,2}$ is closed under fusion.

5.2 $t = 3$

We calculated the following fusion products. The calculations were done again up to level six, apart from the last three identities which were only checked up to level five. We note that the last six identities follow from the previous ones by the associativity and commutativity of the fusion product, and thus provide a consistency check.

$$\begin{aligned}
(\mathcal{V}_{2,1} \otimes \mathcal{V}_{2,1})_f &= \mathcal{V}_{1,1} \oplus \mathcal{V}_{3,1}, \\
(\mathcal{V}_{m,1} \otimes \mathcal{V}_{1,2})_f &= \mathcal{V}_{m,2}, \quad \text{for } m = 1, \dots, 5, \\
(\mathcal{V}_{m,1} \otimes \mathcal{V}_{1,3})_f &= \mathcal{V}_{m,3}, \quad \text{for } m = 1, \dots, 5, \\
(\mathcal{V}_{1,2} \otimes \mathcal{V}_{1,2})_f &= \mathcal{V}_{1,1} \oplus \mathcal{V}_{1,3}, \\
(\mathcal{V}_{1,2} \otimes \mathcal{V}_{1,3})_f &= \mathcal{R}_{1,2}, \\
(\mathcal{V}_{1,3} \otimes \mathcal{V}_{1,3})_f &= \mathcal{R}_{1,1} \oplus \mathcal{V}_{1,3}, \\
(\mathcal{V}_{2,2} \otimes \mathcal{V}_{1,3})_f &= \mathcal{R}_{2,2}, \\
(\mathcal{V}_{2,3} \otimes \mathcal{V}_{1,3})_f &= \mathcal{R}_{2,1} \oplus \mathcal{V}_{2,3}, \\
(\mathcal{V}_{1,2} \otimes \mathcal{R}_{1,1})_f &= \mathcal{R}_{1,2} \oplus \mathcal{V}_{2,3}, \\
(\mathcal{V}_{1,2} \otimes \mathcal{R}_{1,2})_f &= \mathcal{R}_{1,1} \oplus 2\mathcal{V}_{1,3}, \\
(\mathcal{V}_{1,3} \otimes \mathcal{R}_{1,1})_f &= \mathcal{R}_{2,2} \oplus 2\mathcal{V}_{1,3}, \\
(\mathcal{V}_{1,3} \otimes \mathcal{R}_{1,2})_f &= 2\mathcal{R}_{1,2} \oplus \mathcal{V}_{2,3}, \\
(\mathcal{R}_{1,1} \otimes \mathcal{R}_{1,1})_f &= 2\mathcal{R}_{1,1} \oplus \mathcal{R}_{2,2} \oplus \mathcal{V}_{1,3} \oplus \mathcal{V}_{3,3}, \\
(\mathcal{R}_{1,1} \otimes \mathcal{R}_{1,2})_f &= 2\mathcal{R}_{1,2} \oplus \mathcal{R}_{2,1} \oplus 2\mathcal{V}_{2,3}, \\
(\mathcal{R}_{1,2} \otimes \mathcal{R}_{1,2})_f &= 2\mathcal{R}_{1,1} \oplus \mathcal{R}_{2,2} \oplus 4\mathcal{V}_{1,3}.
\end{aligned}$$

We have included the parameters, characterising the generalised highest weight representations, in Table 2. Again, as before, we checked explicitly that all the representations denoted by $\mathcal{R}_{m,n}$ are indeed equivalent representations, *i.e.* have the same characteristic parameter.

(m, n)	$h_{m,n}$	$h_{m-1,t-n}$	$\beta_{m,n}$
$(1, 1)$	0	0	—
$(1, 2)$	$-\frac{1}{4}$	$-\frac{1}{4}$	—
$(2, 1)$	$\frac{7}{4}$	$-\frac{1}{4}$	$\frac{8}{9}$
$(2, 2)$	1	0	-2

Table 2: Representation data of $\mathcal{R}_{m,n}$ for $t = 3$.

The results suggest that we have for all m_1 and m_2

$$(\mathcal{V}_{m_1,1} \otimes \mathcal{V}_{m_2,1})_{\text{f}} = \bigoplus_{m=|m_1-m_2|+1}^{m_1+m_2-1} \mathcal{V}_{m,1}, \quad (5.6)$$

and that for all m

$$\begin{aligned} (\mathcal{V}_{m,1} \otimes \mathcal{V}_{1,2})_{\text{f}} &= \mathcal{V}_{m,2}, \\ (\mathcal{V}_{m,1} \otimes \mathcal{V}_{1,3})_{\text{f}} &= \mathcal{V}_{m,3}, \\ (\mathcal{V}_{m,2} \otimes \mathcal{V}_{1,3})_{\text{f}} &= \mathcal{R}_{m,2}, \\ (\mathcal{V}_{m,3} \otimes \mathcal{V}_{1,3})_{\text{f}} &= \mathcal{V}_{m,3} \oplus \mathcal{R}_{m,1}, \end{aligned} \quad (5.7)$$

where $\mathcal{R}_{m,n}$ is a generalised highest weight representation of the type discussed before, whose characteristic parameter is described by Table 2. This again confirms our conjectures of the previous section.

Together with the above results about the fusion of the generalised highest weight representations, (5.6) and (5.7) are then sufficient to derive the decomposition of all fusion products for $t = 3$, using the associativity and symmetry of the fusion product. We find

$$(\mathcal{V}_{m_1,n_1} \otimes \mathcal{V}_{m_2,n_2})_{\text{f}} = \bigoplus_{m=|m_1-m_2|+1}^{m_1+m_2-1} \left(\bigoplus_{n=|n_1-n_2|+1}^{n_1+n_2-1} \mathcal{V}_{m,n} \right), \quad (5.8)$$

where $(n_1, n_2) = (1, 1); (1, 2); (1, 3)$ or $(2, 2)$. Furthermore

$$\begin{aligned} (\mathcal{V}_{m_1,2} \otimes \mathcal{V}_{m_2,3})_{\text{f}} &= \bigoplus_m \mathcal{R}_{m,2}, \\ (\mathcal{V}_{m_1,3} \otimes \mathcal{V}_{m_2,3})_{\text{f}} &= \bigoplus_m (\mathcal{V}_{m,3} \oplus \mathcal{R}_{m,1}), \\ (\mathcal{V}_{m_1,1} \otimes \mathcal{R}_{m_2,1})_{\text{f}} &= \bigoplus_m \mathcal{R}_{m,1}, \\ (\mathcal{V}_{m_1,1} \otimes \mathcal{R}_{m_2,2})_{\text{f}} &= \bigoplus_m \mathcal{R}_{m,2}, \\ (\mathcal{V}_{m_1,2} \otimes \mathcal{R}_{m_2,1})_{\text{f}} &= \bigoplus_m (\mathcal{R}_{m,2} \oplus \underline{\mathcal{V}_{m-1,3}} \oplus \mathcal{V}_{m+1,3}), \end{aligned}$$

$$\begin{aligned}
(\mathcal{V}_{m_1,2} \otimes \mathcal{R}_{m_2,2})_f &= \bigoplus_m (\mathcal{R}_{m,1} \oplus 2\mathcal{V}_{m,3}) , \\
(\mathcal{V}_{m_1,3} \otimes \mathcal{R}_{m_2,1})_f &= \bigoplus_m (2\mathcal{V}_{m,3} \oplus \underline{\mathcal{R}_{m-1,2}} \oplus \mathcal{R}_{m+1,2}) , \\
(\mathcal{V}_{m_1,3} \otimes \mathcal{R}_{m_2,2})_f &= \bigoplus_m (2\mathcal{R}_{m,2} \oplus \underline{\mathcal{V}_{m-1,3}} \oplus \mathcal{V}_{m+1,3}) , \\
(\mathcal{R}_{m_1,1} \otimes \mathcal{R}_{m_2,1})_f &= \bigoplus_m (2\mathcal{R}_{m,1} \oplus \underline{\mathcal{R}_{m-1,2}} \oplus \mathcal{R}_{m+1,2} \oplus \mathcal{V}_{m,3} \oplus \\
&\quad \underline{\mathcal{V}_{m-2,3}} \oplus \underline{\mathcal{V}_{m,3}} \oplus \mathcal{V}_{m+2,3}) , \\
(\mathcal{R}_{m_1,1} \otimes \mathcal{R}_{m_2,2})_f &= \bigoplus_m (2\mathcal{R}_{m,2} \oplus \underline{\mathcal{R}_{m-1,1}} \oplus \mathcal{R}_{m+1,1} \oplus 2\underline{\mathcal{V}_{m-1,3}} \oplus 2\mathcal{V}_{m+1,3}) , \\
(\mathcal{R}_{m_1,2} \otimes \mathcal{R}_{m_2,2})_f &= \bigoplus_m (2\mathcal{R}_{m,1} \oplus \underline{\mathcal{R}_{m-1,2}} \oplus \mathcal{R}_{m+1,2} \oplus 4\mathcal{V}_{m,3}) ,
\end{aligned}$$

where the underlined terms on the right hand side are absent for $m = 1$ (which can only happen for $m_1 = m_2$), and the doubly underlined term is absent for $m = 1, 2$ (which only happens for $|m_1 - m_2| \leq 1$).

This shows that the set of representations $\mathcal{R}_{m,n}$ and $\mathcal{V}_{m,n}$ is closed under fusion.

5.3 $t \geq 4$

In Section 4 we conjectured (4.1) the decomposition of the fusion product of arbitrary *irreducible* representations for all integer t . The structure of fusion products involving indecomposable representations is more complicated, and *a priori* one cannot exclude the possibility of generating new types of representations. However, we have calculated the products

$$\begin{aligned}
(\mathcal{V}_{1,2} \otimes \mathcal{R}_{1,1})_f &= \mathcal{R}_{1,2} \oplus \mathcal{V}_{2,t}, \\
(\mathcal{V}_{1,2} \otimes \mathcal{R}_{1,n})_f &= \mathcal{R}_{1,n-1} \oplus \mathcal{R}_{1,n+1}, \quad \text{for } 1 < n < t-1, \\
(\mathcal{V}_{1,2} \otimes \mathcal{R}_{1,t-1})_f &= \mathcal{R}_{1,t-2} \oplus 2\mathcal{V}_{1,t}
\end{aligned} \tag{5.9}$$

for $t = 4, \dots, 10$ (up to fusion level $L = 2$), and we conjecture that these identities hold in general. This is then already enough to derive the fusion rules for all representations, using the associativity and symmetry of the fusion product. In particular, we see that the set of representations $\mathcal{R}_{m,n}$ and $\mathcal{V}_{m,n}$ is closed under fusion.

We should note that for all t , the subset of representations $\mathcal{V}_{m,1}$ is closed under fusion, and that the conformal weights of the fields corresponding to $\mathcal{V}_{\text{odd},1}$ are all integral. In particular, the chiral algebra can thus be extended to include these fields. For odd t , there is also another closed subset of representations given by $\mathcal{R}_{\text{odd,odd}}$, $\mathcal{R}_{\text{even,even}}$ and $\mathcal{V}_{\text{odd},t}$.

6 Conclusions

In this paper we have analysed the fusion products of certain representations of the $(1, q)$ models. In particular, we have focused on the fusion products which are not completely reducible, and we have analysed the structure of the indecomposable representations in detail.

We have determined the parameter which characterises the indecomposable representations for a few cases explicitly. We have also studied the fusion rules of these indecomposable representations, and we have seen that a suitable extended set of representations closes under fusion.

This extended set of representations is the smallest set of representations (containing all allowed irreducible representations) which closes under fusion. It is therefore the natural analogue of a “minimal model” for the $(1, q)$ case, and it seems plausible that it should give rise to a consistent conformal field theory. In order to check whether this is indeed the case, it would be necessary to determine the (chiral) correlation functions, and to analyse whether there exist suitable chiral-antichiral combinations which define a crossing-symmetric and local theory.

In the same spirit, it would be interesting to analyse these representations from the point of view of an enlarged chiral symmetry algebra. In particular, one could consider the extension of the Virasoro algebra by three fields of conformal weight $(2q - 1)$ (corresponding to the field $(3, 1)$), the so-called “triplet algebra” [18]. It might then happen that the infinitely many representations in the extended set (which close under fusion with respect to the Virasoro algebra) form finitely many representations of the triplet algebra (which close under fusion with respect to the triplet algebra). The corresponding theory would then be a “rational logarithmic” conformal field theory. This is currently under investigation.

We have also introduced a new algorithm which allows a finite analysis of fusion products up to any given level. The algorithm can in principle be applied to any chiral theory, but we have only been able to show that it terminates for the case of the Virasoro algebra. It would be interesting to show that this restriction is not necessary.

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Appendix: An example

As an example, we present here the matrices obtained for the fusion product

$$(\mathcal{V}_{2,1} \otimes \mathcal{V}_{1,2})_f = \mathcal{R}_{2,1}$$

for $t = 2$ up to level four. The canonical bases for $\mathcal{R}_{2,1}^n$ up to level $L = 4$ are composed of four blocks consisting of lexicographically ordered monomials on the states ξ, ϕ, ψ and ρ ,

This matrix explicitly exhibits the Jordan cell structure between the blocks $[\phi]$ and $[\psi]$. According to our discussion of Section 4, the matrices for the positive Virasoro modes have the general structure

$$\begin{matrix} & [\xi] & [\phi] & [\psi] & [\rho] \\ \begin{matrix} [\xi] \\ [\phi] \\ [\psi] \\ [\rho] \end{matrix} & \begin{pmatrix} * & 0 & * & 0 \\ * & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \end{matrix}$$

reflecting, for example, that states in the block $[\xi]$ cannot be mapped to states in the blocks $[\psi]$ and $[\rho]$, and similarly in the other cases. In more detail, we find

$$L_1 = \begin{pmatrix} \begin{array}{c|c|c|c} & [\xi] & [\phi] & [\psi] \\ \hline [\xi] & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & 4 & \cdot & \cdot \\ \cdot & \cdot & 5 & 3 \\ \hline \cdot & 3 & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & 2 & \cdot & \cdot \end{array} & \begin{array}{cccc} \boxed{-1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 & 3/2 \\ \hline \cdot & 1 & \cdot & \cdot \end{array} \\ \hline [\phi] & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot \\ \cdot & \cdot & 4 & 8 \\ \hline \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \hline \cdot & 2 & \cdot & \cdot \end{array} \\ \hline [\psi] & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & 3 & \cdot & \cdot \\ \cdot & \cdot & 4 & 8 \\ \hline \cdot & \cdot & \cdot & 5 & 2 & 3 \\ \cdot & \cdot & \cdot & \cdot & 4 & 6 \\ \hline \cdot & \cdot & \cdot & 3 & \cdot & \cdot \end{array} \\ \hline [\rho] & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & 6 & \cdot & \cdot \\ \cdot & \cdot & 3 & 14 \end{array} \end{array} \end{pmatrix},$$

where the boxed entry is the characteristic parameter, $\beta_{2,1} = -1$, of $\mathcal{R}_{2,1}$. The other matrices are:

$$L_2 = \begin{matrix} & [\xi] & & [\phi] & & [\psi] & & [\rho] \\ \begin{matrix} [\xi] \\ [\phi] \\ [\psi] \\ [\rho] \end{matrix} & \left(\begin{array}{cccc|cccc|cccc|cccc} \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 6 & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & -3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 5 & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{21}{2} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{3}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & 7 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & 10 & 14 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 5 & \cdot & \cdot & \cdot & 11 & 18 \end{array} \right) \end{matrix}$$

$$L_3 = \begin{matrix} & [\xi] & & & [\phi] & & & [\psi] & & & [\rho] \\ \begin{matrix} [\xi] \\ [\phi] \\ [\psi] \end{matrix} & \left(\begin{array}{ccc|ccc|ccc|ccc} \cdot & \cdot & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -5 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 7 & 15 & \cdot & \cdot & \cdot & 2 & 10 & \cdot & \cdot & \cdot & 6 & 5 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & 10 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 7 & 8 & 15 & \cdot & \cdot & \cdot & \cdot \end{array} \right) \end{matrix}$$

$$L_4 = \begin{array}{c} [\xi] \qquad \qquad \qquad [\phi] \qquad \qquad \qquad [\psi] \qquad \qquad \qquad [\rho] \\ \begin{array}{c|c|c|c} \begin{array}{cccc} \cdot & \cdot & \cdot & -10 \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & -7 \end{array} & \begin{array}{cccc} -18 & \cdot & \cdot & \cdot \end{array} \\ \hline \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} \cdot & \cdot & \cdot & \cdot \end{array} & \begin{array}{cccc} -2 & 14 & 18 & \cdot \end{array} \\ \hline \end{array} \end{array} \begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \cdot \end{array} \end{array} \Bigg) \\ \uparrow$$

Here we have indicated by an arrow the column corresponding to $L_n \psi = 0$ for $n > 1$.

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